The effect of uniform distortion on weak homogeneous turbulence

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The behaviour of weak homogeneous turbulence subjected to a uniform distortion is studied by means of linearized equations that include the effect of inertial interaction between the mean flow and the turbulent fluctuations and of viscous dissipation. Three particular cases of distortion are investigated in varying degrees of detail; those of uniform rotation, uniform shear, and uniform irrotational distortion. For the first two, the total energy associated with the turbulence is found to decay, but for the third it is found in general to increase without limit. A general solution in terms of the spectrum functions for the restricted case of irrotational distortions is given, particular consideration being paid to the asymptotic limits approached as the distortions (not necessarily the rates of distortion) become large. These show that for an arbitrary initial turbulent spectrum the asymptotic rates of growth are functions of the precise geometrical nature of the distortion, of an integral parameter of the initial spectrum function, of a Reynolds number based on the rate of distortion and the length scale of the initial turbulence, and of the contraction. A particular example, that of initially isotropic turbulence in its final period of decay subjected to an axisymmetric distortion, is worked out in detail and the solution (described in terms of mean turbulent intensities) displayed graphically for several values of the Reynolds number.

1. Introduction

When a weak homogeneous turbulent velocity field is uniformly distorted we may use a suitably linearized form of the Navier–Stokes equations to determine the effect of the distortion on the turbulent field. We suppose that the velocity field is the sum of a steady uniform distortion and of a small turbulent fluctuation. By neglecting quadratic terms in the turbulent velocities, linear equations are obtained. By well-known methods these can be converted into equations for the spectrum functions in wave-number space, and these in turn are, in certain particular cases, simply soluble. These equations preserve in some degree the nature of the full Navier–Stokes equations of motion, in that inertial forces—here the product of the interaction between steady flow and turbulence—pressure forces and viscous forces are all included.

The effect of the interaction terms alone has been considered by Batchelor & Proudman (1954) (B & P hereafter), who suppose the strain to be sudden and employ a Lagrangian approach in order to evaluate the effect of distortion on any particular small fluid element. A consideration of the simultaneous effects of inertial interaction and viscous dissipation suggests an Eulerian treatment such as has been outlined above. It is found that the results of B & P are obtained more readily by means of the latter treatment, while several other interesting results concerning the asymptotic states achieved after long times of mean strain are derived. The problem has also been discussed by Ribner & Tucker (1952); they survey some of the experimental results that are relevant, sufficiently at least to show that the particular linearization chosen is a suitable approximation to certain wind-tunnel arrangements.

The notation used throughout follows that of Proudman & Reid (1954) (P & R). Mean values or averages, denoted by an overbar, are assumed to be ensemble averages. This avoids any difficulties that might arise if spatial averages were used. The validity of Fourier transformation has been discussed by Batchelor (1953).

2. Dynamical equations

We consider the uniform strain to be caused by a mean motion

$$U_i = C_{ij} x_j, \tag{2.1}$$

where C_{ij} are constants independent of position. For simplicity we shall take them to be independent of time also, though this is not essential; the analysis can be carried out equally well for the case

$$C_{ij} = C_{ij}(t). \tag{2.2}$$

We consider a turbulent velocity field u_i to be superposed on the mean strain U_i such that the velocity becomes the sum

$$v_i = U_i + u_i \tag{2.3}$$

and the pressure p is given by

$$p/\rho = P + \varpi, \tag{2.4}$$

ρ being the uniform density, P the mean component and ϖ the fluctuating component of the reduced pressure. The Navier–Stokes equation and the continuity condition are

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 v_i}{\partial x_j \partial x_j}, \quad \frac{\partial v_i}{\partial x_i} = 0, \tag{2.5}$$

where ν is the kinematic viscosity. By taking mean values we get

$$U_j \frac{\partial U_i}{\partial x_j} + \frac{\partial P}{\partial x_i} - \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j} = 0, \quad \frac{\partial U_i}{\partial x_i} = 0, \tag{2.6}$$

since $\partial U_i/\partial t$, $\overline{u_i}$, $\overline{\varpi}$ are by definition zero; $\partial(\overline{u_i u_j})/\partial x_k$ we take to be zero, because we suppose further that the turbulent field is homogeneous. It can readily be shown that if at any stage the turbulence is homogeneous then it will remain so, and thus the postulate of a homogeneous turbulent field is self-consistent.

The equations for the turbulent components, u_i, become

$$\frac{\partial u_i}{\partial t} = -U_j\frac{\partial u_i}{\partial x_j} - u_j\frac{\partial U_i}{\partial x_j} - u_j\frac{\partial u_i}{\partial x_j} - \frac{\partial \varpi}{\partial x_i} - \nu\frac{\partial^2 u_i}{\partial x_j \partial x_j}, \tag{2.7}$$

$$\frac{\partial u_i}{\partial x_i} = 0. \tag{2.8}$$

Because the turbulence is assumed to be weak, we are justified in neglecting the terms $u_j(\partial u_i/\partial x_j)$ and the equation (2.7) can be linearized to give

$$\frac{\partial u_i}{\partial t} + U_j\frac{\partial u_i}{\partial x_j} + u_j\frac{\partial U_i}{\partial x_j} + \frac{\partial \varpi}{\partial x_i} - \nu\frac{\partial^2 u_i}{\partial x_j \partial x_j} = 0. \tag{2.9}$$

We now define the following velocity product and pressure-velocity product mean values

$$\left.\begin{aligned} R_{ij}(\mathbf{r},t) &= \overline{u_i(\mathbf{x},t)\,u_j(\mathbf{x}+\mathbf{r},t)},\\ P_i(\mathbf{r},t) &= \overline{\varpi(\mathbf{x},t)\,u_i(\mathbf{x}+\mathbf{r},t)},\\ W_{ij}(\mathbf{r},t) &= \overline{\omega_i(\mathbf{x},t)\,\omega_j(\mathbf{x}+\mathbf{r},t)},\end{aligned}\right\} \tag{2.10}$$

where $\omega_i(\mathbf{x},t)$ represents the vorticity associated with the velocity u_i and is given by

$$\omega_i = \epsilon_{ijk}\frac{\partial u_k}{\partial x_j}. \tag{2.11}$$

The functions on the left-hand side are written as functions of $\mathbf{r}$ and t only, since the original postulate of homogeneity removes the dependence on $\mathbf{x}$. We also define the Fourier transforms of the functions (2.10) as

$$\left.\begin{aligned} \Phi_{ij}(\mathbf{k},t) &= (2\pi)^{-3}\int R_{ij}(\mathbf{r},t)\,e^{-i\mathbf{k}.\mathbf{r}}\,d\mathbf{r},\\ \Pi_i(\mathbf{k},t) &= i(2\pi)^{-3}\int P_i(\mathbf{r},t)\,e^{-i\mathbf{k}.\mathbf{r}}\,d\mathbf{r},\\ \Omega_{ij}(\mathbf{k},t) &= (2\pi)^{-3}\int W_{ij}(\mathbf{r},t)\,e^{-i\mathbf{k}.\mathbf{r}}\,d\mathbf{r}.\end{aligned}\right\} \tag{2.12}$$

The many symmetry properties of the functions (2.10) and (2.12) and the consequences of the continuity equation are all given in P & R and are not repeated here.

In terms of these functions the equation (2.9) leads to the following dynamical equations for $R_{ij}(\mathbf{r},t)$

$$\begin{aligned}\frac{\partial}{\partial t}R_{ij}(\mathbf{r},t) = &-C_{ik}R_{kj}(\mathbf{r},t) - C_{jk}R_{ik}(\mathbf{r},t) - C_{kl}r_l\frac{\partial}{\partial r_k}R_{ij}(\mathbf{r},t)\\ &+\frac{\partial}{\partial r_i}P_j(\mathbf{r},t) + \frac{\partial}{\partial r_j}P_i(\mathbf{r},t) + 2\nu\frac{\partial^2}{\partial r_l \partial r_l}R_{ij}(\mathbf{r},t).\end{aligned} \tag{2.13}$$

We may now take Fourier transforms of the equations (2.13) and eliminate the terms $\Pi_i(\mathbf{k},t)$ by contraction and resubstitution (as in P & R) to obtain the relevant dynamical equations for the spectrum functions

$$\begin{aligned}\frac{\partial \Phi_{ij}(\mathbf{k},t)}{\partial t} = C_{lk}\bigg\{&\left(2\frac{k_i k_l}{k^2} - \delta_{il}\right)\Phi_{kj}(\mathbf{k},t) + \left(2\frac{k_j k_l}{k^2} - \delta_{jl}\right)\Phi_{ik}(\mathbf{k},t)\\ &+ k_l\frac{\partial}{\partial k_k}\Phi_{ij}(\mathbf{k},t)\bigg\} - 2\nu k^2\Phi_{ij}(\mathbf{k},t).\end{aligned} \tag{2.14}$$

Although well-known methods are available for solving this set of nine first-order simultaneous equations, it is not possible to display the general solution in neat tensorial form for arbitrary C_{lk} and arbitrary initial values of $\Phi_{ij}(\mathbf{k})$. Indeed, in most cases, particular solutions become excessively complex.

One result concerning a general distortion is worth noting. Though the mean strain may be split into a uniform rotation plus a uniform irrotational distortion, the solution of the equations (2.14) (and equivalently for the inviscid case, or sudden strain approximation) is not such that the effects of rotation and irrotational distortion may be completely separated. In their sudden strain approximation, B & P suppose that the effect of rotation may be neglected; an inspection of equations (2.14) shows that this cannot in fact be true.

Three particular cases have been examined in varying degrees of detail, those of uniform rotation, uniform shear and uniform irrotational distortion. The asymptotic form of the solution as $t \to \infty$, when the spectrum is supposed known at some initial instant, provides the most interesting information and this aspect of the solution is the one to which most attention will be given; for this limit tells us whether or not the turbulent field extracts more energy from the mean strain than is dissipated by viscosity. It also tells us how the turbulent field becomes oriented. If the asymptotic solution leads to a continuous increase in turbulent intensity then ultimately the linearization that we have adopted must fail, and inertial interactions within the turbulent field will become important. Indeed, experience shows that these interactions suffice to limit the growth in turbulent intensity. However, if we suppose the initial turbulence to be sufficiently weak, the effect of interactions is only felt after the asymptotic form has been substantially achieved.

These results have some connexion with stability theory: the arbitrary spatially bounded initial disturbance of stability theory is replaced in our case by an initial spectrum of (homogeneous) turbulence. Since in practice the arbitrary initial disturbance of stability theory is usually chosen to be a sinusoidal wave-like motion, the connexion becomes reasonably close. The presence of boundaries (or equivalently, the condition of zero disturbance at infinity) in the former and the absence of boundaries in the latter make the treatments complimentary.

Uniform rotation. This case can be represented by $C_{23} = -C_{32} =$ constant, all other $C_{ij} = 0$. A full solution of the equations (2.14) for this choice of C_{ij} has not been obtained because of their apparent intractability. However, it can readily be shown that the total kinetic energy associated with the turbulent field, $\int \Phi_{ii}(\mathbf{k}, t)\, d\mathbf{k}$, decays in time, in much the same way as it would in the absence of the mean rotation. By taking the inviscid case, we find that the total kinetic energy remains constant. Although these statements are rather crude, they represent all the information that has been obtained; in particular, no information about orientation along or perpendicular to the axis of rotation was derived.

Uniform shear. This case can be represented by $C_{12} =$ constant, all other $C_{ij} = 0$. A complete solution has been obtained, but this is much too complicated to present here. As in the case of uniform rotation the relevant result is that the total energy ultimately decays with time. This result depends, it must be pointed out, on the asymptotic form of the initial spectrum, $\Phi_{ij}(\mathbf{k}, 0)$, near $k = 0$, which, as Batchelor & Proudman (1956) have shown, must obey the relation

$$\Phi_{ij}(\mathbf{k}, 0) = \frac{C_{lmnp}}{4\pi^2}\left(\delta_{il} - \frac{k_i k_l}{k^2}\right)\left(\delta_{jm} - \frac{k_j k_m}{k^2}\right) k_n k_p + O(k^3 \ln k). \qquad (2.15)$$

The precise form of (2.15) makes it very difficult to decide whether any significant orientation is taking place within the decaying system. However, the detailed asymptotic structure of the turbulent field would only be decisively important if the turbulent intensity were increasing, which is not the case. It is perhaps worth noting that the result we have obtained is consistent with the established result that (plane) Couette motion is stable to all small disturbances; this stability is evidently not dependent on the presence of boundaries.

Uniform irrotational distortion. This proves to be the most interesting case of the three and the solution is presented in some detail in the following section. The general solution for a known initial spectrum is found to depend on the precise geometrical form of the distortion. If we define an expansion ratio, c, as the total relative amount of strain that has taken place along the principal axis of maximum rate of expansion, then the asymptotic behaviour of the total kinetic energy associated with the turbulent field can be expressed as a power of c, the index of this power lying between 0 and 1. For distortions in which expansion takes place along two of the principal axes of strain, the index is always unity, but for distortions in which contraction takes place along two of the principal axes of strain, the growth in energy is proportional to the ratio of the two contractions, respectively. Thus for axisymmetric distortion involving expansion along the axis of symmetry the index is zero, and the total energy tends to a constant. The coefficient of proportionality depends on the rate of straining, and involves a rather complicated integral parameter of the initial turbulent spectrum. The special case of isotropic turbulence is treated as an example. This allows the integral parameter to be expressed in a rather simpler form.

3. General solution for irrotational distortion. Asymptotic limit for large distortions

The restricted case of a uniform irrotational distortion will now be considered in detail; the time-dependent solution for an initially known weak turbulent field that is subjected to a uniform mean strain will be developed.

The axes of the co-ordinate system can be chosen to lie along the principal axes of strain such that

$$C_{11} = \frac{1}{T}, \quad C_{22} = -\frac{\alpha}{T}, \quad C_{33} = \frac{\alpha - 1}{T}. \qquad (3.1)$$

T represents a characteristic time describing the rate of distortion and α is a geometrical parameter such that $-1 \leqslant \alpha \leqslant \frac{1}{2}$. Let us define an 'expansion ratio' c by the relation

$$C = e^{t/T}, \qquad (3.2)$$

whence

$$\frac{\partial}{\partial t} \equiv \frac{c}{T}\frac{\partial}{\partial c}, \qquad (3.3)$$

c representing the extension along the principal axis of maximum extension. The equations (2.14) become

$$c\frac{\partial}{\partial c}\Phi_{ij}(\mathbf{k},c)+\lambda^{(i)}\left[\left(\delta_{il}+\frac{2k_ik_l}{k^2}\right)\Phi_{lj}(\mathbf{k},c)+\left(\delta_{jl}-\frac{2k_jk_l}{k^2}\right)\Phi_{li}(k,c)\right.$$
$$\left.-k_l\frac{\partial}{\partial k_l}\Phi_{ij}(\mathbf{k},c)\right]+2\nu Tk^2\Phi_{ij}(\mathbf{k},c)=0, \quad (3.4)$$

where
$$\lambda^{(1)},\lambda^{(2)},\lambda^{(3)}=1,\,-\alpha,\alpha-1. \quad (3.5)$$

The equivalent equations for the vorticity spectrum functions $\Omega_{ij}(\mathbf{k},c)$ are then

$$c\frac{\partial}{\partial c}\Omega_{ij}(\mathbf{k},c)-(\lambda^{(i)}+\lambda^{(j)})\,\Omega_{ij}(\mathbf{k},c)-\lambda^{(l)}k_l\frac{\partial}{\partial k_l}\Omega_{ij}(\mathbf{k},c)+2\nu Tk^2\Omega_{ij}(\mathbf{k},c)=0, \quad (3.6)$$

where
$$k^2\Phi_{ji}(\mathbf{k},c)=\left(\frac{k^2\delta_{ij}-k_ik_j}{k^2}\right)\Omega_{ll}(\mathbf{k},c)-\Omega_{ij}(\mathbf{k},c). \quad (3.7)$$

To solve (3.6) we observe that the equation

$$\left(c\frac{\partial}{\partial c}-\lambda^{(l)}k_l\frac{\partial}{\partial k_l}\right)F_{ij}(\mathbf{k},c)=0 \quad (3.8)$$

has a solution
$$F_{ij}(\mathbf{k},c)=F_{ij}^{(0)}(\boldsymbol{\chi}), \quad (3.9)$$

where
$$\boldsymbol{\chi}=(ck_1,c^{-\alpha}k_2,c^{\alpha-1}k_3). \quad (3.10)$$

$F_{ij}^{(0)}(\boldsymbol{\chi})$ is an arbitrary function, except when $\alpha=0$. (The case $\alpha=0$ needs special treatment.) Next we observe that the equation (3.6) with $\nu\equiv 0$ has a solution

$$\Omega_{ij}^{(\nu=0)}(\mathbf{k},c)=c^{\lambda^{(i)}+\lambda^{(j)}}\Omega_{ij}^{(\nu=0)(0)}(\boldsymbol{\chi}). \quad (3.11)$$

Finally, we deduce that the full solution of (3.6) is given by

$$\Omega_{ij}(\mathbf{k},c)=c^{\lambda^{(i)}+\lambda^{(j)}}\Omega_{ij}^{(0)}(\boldsymbol{\chi})\,B(\boldsymbol{\chi})/B(\mathbf{k}), \quad (3.12)$$

where
$$B(\mathbf{k})=\exp\left\{-\nu T\left(k_1^2-\frac{1}{\alpha}k_2^2-\frac{1}{1-\alpha}k_3^2\right)\right\}. \quad (3.13)$$

The function $\Omega_{ij}^{(0)}(\boldsymbol{\chi})$ can be obtained since $\Omega_{ij}(\mathbf{k},c)$ is known at the initial instant, $c=1$. The form for $\Phi_{ij}(\mathbf{k},c)$ follows from relation (3.7). The intermediate solution (3.11) is just the one obtained by B & P.

For a general initial vorticity spectrum function, given by $\Omega_{ij}^{(0)}(\boldsymbol{\chi})$, the three mean square turbulent intensities, given by

$$\overline{u_1^2}=\iiint\Phi_{11}(\mathbf{k},c)\,d\mathbf{k},$$
$$\overline{u_2^2}=\iiint\Phi_{22}(\mathbf{k},c)\,d\mathbf{k}$$

and
$$\overline{u_3^2}=\iiint\Phi_{33}(\mathbf{k},c)\,d\mathbf{k},$$

can in principle be evaluated for all values of c thus providing a measure both of the absolute growth or decay of the turbulent intensity and of its relative orientation.

The form of the solution for the asymptotic limit $c \to \infty$ will now be considered, since this may be taken to indicate the long-term effect of the straining motion on the turbulence. It also allows the integrals to be simplified sufficiently for certain general results to be derived.

It proves convenient to discuss the following distortions separately:

(A) $0 < \alpha \leqslant \frac{1}{2}$; two principal axes of contraction.

(B) $\alpha = 0$; 'constant area' deformation.

(C) $-1 \leqslant \alpha < 0$; two principal axes of expansion.

The calculations involved in arriving at the asymptotic limits given below are not included in detail since they are very largely straightforward. However, reference will be made to those points in the calculation at which particular properties—derived from the continuity condition or the equations of motion—of the initial spectrum function are employed.

(A) $0 < \alpha < \frac{1}{2}$. (The case $\alpha = \frac{1}{2}$ is similar but requires slightly more elaborate treatment.)

$$\begin{aligned}\underset{c\to\infty}{\mathrm{Lt}}\ \overline{u_1^2} &= \underset{c\to\infty}{\mathrm{Lt}} \iiint \Phi_{11}(\mathbf{k}, c)\, d\mathbf{k} \\ &= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{l_1^2}{\{(l_1^2/c^2)+l_2^2+l_3^2\}} \frac{\Omega_{11}^{(0)}(l_1, c^{-\alpha}l_2, c^{\alpha-1}l_3)}{c} \\ &\qquad \times \exp\left\{-\nu T\left(l_1^2+\frac{l_2^2}{\alpha}+\frac{l_3^2}{1-\alpha}\right)\right\} dl_1\, dl_2\, dl_3 + \text{smaller terms},\end{aligned}$$

from (3.12), (3.13) and (3.7), where $\mathbf{l}$ is a variable of integration related linearly to $\mathbf{k}$.

Before we can further simplify this expression we must make use of the result that

$$\Omega_{11}^{(0)}(l_1, 0, 0) \equiv 0. \tag{3.14}$$

[See Batchelor (1953), p. 28.] If we suppose that the function $\Omega_{11}^{(0)}(l_1, c^{-\alpha}l_2, c^{1-\alpha}\, l_3)$ can be expanded in the form*

$$\Omega_{11}^{(0)}(l_1, c^{-\alpha}l_2, c^{1-\alpha}l_3) = \tfrac{1}{2}c^{-2\alpha}l_2^2 \frac{\partial^2\Omega_{11}^{(0)}}{\partial k_2^2}(l_1, 0, 0) + \tfrac{1}{2}c^{2\alpha-2}l_3^2 \frac{\partial^2\Omega_{11}^{(0)}}{\partial k_3^2}(l_1, 0, 0) + \text{smaller terms}, \tag{3.15†}$$

the lower order terms not being present because of (3.14) or because of the conditions of symmetry that must be satisfied by the spectrum function, then we may write

$$\begin{aligned}\overline{u_1^2} &\to -\iiint_{-\infty}^{\infty} \frac{c^{-1-2\alpha}l_1^2 l_2^2}{2\{(l_1^2/c^2)+l_2^2+l_3^2\}^2} \frac{\partial^2\Omega_{11}^{(0)}}{\partial k_2^2}(l_1, 0, 0) \exp\left\{-\nu T\left(l_1^2+\frac{l_2^2}{\alpha}+\frac{l_3^2}{1-\alpha}\right)\right\} dl_1\, dl_2\, dl_3 \\ &\qquad + O(c^{-1-2\alpha}\ln c) \\ &\to -\frac{\pi}{4}\iint_{-\infty}^{\infty} c^{-1-2\alpha} \frac{l_1^2 l_2^2}{\{(l_1^2/c^2)+l_2^2\}^{\frac{3}{2}}} \frac{\partial^2\Omega_{11}^{(0)}}{\partial k_2^2} \exp\left\{-\nu T\left(l_1^2+\frac{l_2^2}{\alpha}\right)\right\} dl_1\, dl_2 + O(c^{-1-2\alpha}\ln c) \\ &\to O(c^{-1-2\alpha}\ln c).\end{aligned} \tag{3.16}$$

* This form is certainly valid near the origin in l-space—where difficulties might otherwise be expected—using Batchelor & Proudman (1956), equation (5.31).

† $\dfrac{\partial^2\Omega_{11}^{(0)}}{\partial k_2^2}(l_1, 0, 0)$ denotes $\dfrac{\partial^2\Omega_{11}^{(0)}(\mathbf{k})}{\partial k_2^2}$ evaluated at $(l_1, 0, 0)$.

In a similar fashion we find that

$$\overline{u_2^2} \to \iiint_{-\infty}^{\infty} \frac{c^{-1-2\alpha}}{2} \frac{(l_1^2+c^2l_3^2)\,l_2^2}{\{(l_1^2/c^2)+l_2^2+l_3^2\}^2} \frac{\partial^2\Omega_{11}^{(0)}}{\partial k_2^2}(l_1,0,0) \times \exp\left\{-\nu T\left(l_1^2+\frac{l_2^2}{\alpha}+\frac{l_3^2}{1-\alpha}\right)\right\} dl_1\,dl_2\,dl_3 + \text{smaller terms}$$
$$\to \frac{\pi\alpha}{4\nu T} c^{1-2\alpha} \int_{-\infty}^{\infty} \frac{\partial^2\Omega_{11}^{(0)}}{\partial k_2^2}(l_1,0,0)\,e^{-\nu T l_1^2}\,dl_1 + \text{smaller terms}, \tag{3.17}$$

and that

$$\overline{u_3^2} \to \iiint_{-\infty}^{\infty} \frac{c^{-1-2\alpha}}{2} \frac{(l_1^2+c^2l_2^2)\,l_2^2}{\{(l_1^2/c^2)+l_2^2+l_3^2\}^2} \frac{\partial^2\Omega_{11}^{(0)}}{\partial k_2^2}(l_1,0,0) \times \exp\left\{-\nu T\left(l_1^2+\frac{l_2^2}{\alpha}+\frac{l_3^2}{1-\alpha}\right)\right\} dl_1\,dl_2\,dl_3$$
$$\to \frac{\pi\alpha}{4\nu T} c^{1-2\alpha} \int_{-\infty}^{\infty} \frac{\partial^2\Omega_{11}^{(0)}}{\partial k_2^2}(l_1,0,0)\,e^{-\nu T l_1^2}\,dl_1 + \text{smaller terms}. \tag{3.18}$$

We see then that the total kinetic energy behaves as $c^{1-2\alpha}$—which may be written $c^{1-\alpha}/c^{\alpha}$, that is, as the ratio of the larger contraction to the smaller contraction—and that it becomes wholly oriented axisymmetrically in the plane perpendicular to the principal axis of expansion. The asymptotic limits (3.17) and (3.18) for $\overline{u_2^2}$ and $\overline{u_3^2}$ have been expressed in terms of an integral parameter

$$\int_{-\infty}^{\infty} \frac{\partial^2\Omega_{11}^{(0)}}{\partial k_2^2}(l_1,0,0)\,e^{-\nu T l_1^2}\,dl_1$$

which may be assumed known if the initial turbulence is completely specified.

The case $\alpha = \frac{1}{2}$ leads to the corresponding results

$$\overline{u_1^2} \to O(c^{-2}\ln c), \tag{3.16'}$$

$$\overline{u_2^2}, \overline{u_3^2} \to \frac{\pi}{4\nu T} \int_{0}^{\infty} \frac{\partial^2\Omega_{11}^{(0)}}{\partial k_2^2}(l_1,0,0)\,e^{-\nu T l_1^2}\,dl_1 \tag{3.17'–3.18'}$$

which is a constant, since

$$\frac{\partial^2\Omega_{11}^{(0)}}{\partial k_2^2}(l_1,0,0) \equiv \frac{\partial^2\Omega_{11}^{(0)}}{\partial k_3^2}(l_1,0,0).$$

(B) $\alpha = 0$.

We find that

$$\overline{u_1^2} \to O(c^{-1}\ln c), \tag{3.19}$$

$$\overline{u_2^2} \to \iiint_{-\infty}^{\infty} \frac{(l_1^2+c^2l_3^2)\,\Omega_{11}^{(0)}(l_1, l_2(2\ln c)^{-\frac{1}{2}}, 0)}{\left(\dfrac{l_1^2}{c^2}+\dfrac{l_2^2}{2\ln c}+l_3^2\right)^2} \frac{\exp\{-\nu T(l_1^2+l_2^2+l_3^2)\}}{c(2\ln c)^{\frac{1}{2}}}\,dl_1\,dl_2\,dl_3$$
$$\to \frac{\pi}{8\nu T}\frac{c}{\ln c} \int_{-\infty}^{\infty} \frac{\partial^2\Omega_{11}^{(0)}}{\partial k_2^2}(l_1,0,0)\,e^{-\nu T l_1^2}\,dl_1, \tag{3.20}$$

$$\overline{u_3^2} \to \iiint_{-\infty}^{\infty} \frac{(l_1^2+c^2(2\ln c)^{-1}l_2^2)\,\Omega_{11}^{(0)}(l_1, l_2(2\ln c)^{-\frac{1}{2}}, 0)}{\left(\dfrac{l_1^2}{c^2}+\dfrac{l_2^2}{2\ln c}+l_3^2\right)^2} \frac{\exp\{-\nu T(l_1^2+l_2^2+l_3^2)\}}{c(2\ln c)^{\frac{1}{2}}}\,dl_1\,dl_2\,dl_3$$
$$\to \frac{\pi}{8\nu T}\frac{c}{\ln c} \int_{-\infty}^{\infty} \frac{\partial^2\Omega_{11}^{(0)}}{\partial k_2^2}(l_1,0,0)\,e^{-\nu T l_1^2}\,dl_1. \tag{3.21}$$

The asymptotic limits (3.20) and (3.21) that are $O(c(\ln c)^{-1})$ do not in fact constitute a discontinuity in behaviour of $\overline{u_2^2}$, $\overline{u_3^2}$ at $\alpha = 0$ because of the factor α in (3.17) and (3.18) [and a similar disguised factor in (3.23) and (3.24) below].

(C) $-1 < \alpha < 0$. [The case $\alpha = -1$ is similar, but, like the case $\alpha = \frac{1}{2}$, it requires slightly more elaborate treatment.]

Again using the relations (3.12), (3.13) and (3.7) we obtain the limits

$$\overline{u_1^2} \to \iiint_{-\infty}^{\infty} \frac{(l_2^2 + c^{-2\alpha} l_3^2)\,\Omega_{22}^{(0)}(l_1, l_2, 0) - l_1^2 \Omega_{11}^{(0)}(l_1, l_2, 0)}{\left(\dfrac{l_1^2}{c^2} + \dfrac{l_2^2}{c^{-2\alpha}} + l_3^2\right)^2} \times \frac{\exp\left\{-\nu T\left(l_1^2 + \dfrac{l_2^2}{-\alpha} + \dfrac{l_3^2}{1-\alpha}\right)\right\}}{c^{1-\alpha}}\, dl_1\, dl_2\, dl_3 = O(c^{-2\alpha-1} \ln c^{1+\alpha}), \quad (3.22)$$

$$\overline{u_2^2} \to \iiint_{-\infty}^{\infty} \frac{l_3^2 \Omega_{11}^{(0)}(l_1, l_2, 0)}{\left(\dfrac{l_1^2}{c^2} + \dfrac{l_2^2}{c^{-2\alpha}} + l_3^2\right)^2} \exp\left\{-\nu T\left(l_1^2 + \frac{l_2^2}{-\alpha} + \frac{l_3^2}{1-\alpha}\right)\right\} c^{1+\alpha}\, dl_1\, dl_2\, dl_3$$

$$\to \frac{\pi}{2} c \iint_{-\infty}^{\infty} \frac{\Omega_{11}^{(0)}(l_1, l_2, 0)}{|l_2|} \exp\left\{-\nu T\left(l_1^2 + \frac{l_2^2}{-\alpha}\right)\right\} dl_1\, dl_2, \quad (3.23)$$

and an identical asymptotic expression for $\overline{u_3^2}$. The relevant integral parameter that appears in (3.23) is not so simple as the one obtained in (3.17) and (3.18). The convergence of the integral is assured because of the l_2^2 behaviour of $\Omega_{11}^{(0)}$ near $l_2 = 0$ as given by (2.15). We see that the total kinetic energy grows as c and that once again it becomes oriented axisymmetrically in the plane perpendicular to the axis of greatest expansion. For $\alpha \leqslant -\frac{1}{2}$ the mean square component $\overline{u_1^2}$ does increase as $c \to \infty$, whereas for $\alpha > -\frac{1}{2}$, it tends asymptotically to zero.

The case $\alpha = -1$ leads to the solution

$$\left.\begin{aligned}
\overline{u_1^2} &\to \frac{\pi}{2} c \iint_{-\infty}^{\infty} \left[\frac{l_2^2 \Omega_{22}^{(0)} - l_1^2 \Omega_{11}^{(0)}}{(l_1^2 + l_2^2)^{\frac{3}{2}}} + \frac{\Omega_{22}^{(0)}}{(l_1^2 + l_2^2)^{\frac{1}{2}}}\right] \exp\{-\nu T(l_1^2 + l_2^2)\}\, dl_1\, dl_2,\\
\overline{u_2^2} &\to \frac{\pi}{2} c \iint_{-\infty}^{\infty} \left[\frac{l_1^2 \Omega_{11}^{(0)} - l_2^2 \Omega_{22}^{(0)}}{(l_1^2 + l_2^2)^{\frac{3}{2}}} + \frac{\Omega_{11}^{(0)}}{(l_1^2 + l_2^2)^{\frac{1}{2}}}\right] \exp\{-\nu T(l_1^2 + l_2^2)\}\, dl_1\, dl_2,\\
\overline{u_3^2} &\to \frac{\pi}{2} c \iint_{-\infty}^{\infty} \left[\frac{\Omega_{11}^{(0)} + \Omega_{22}^{(0)}}{(l_1^2 + l_2^2)^{\frac{1}{2}}}\right] \exp\{-\nu T(l_1^2 + l_2^2)\}\, dl_1\, dl_2,
\end{aligned}\right\} \quad (3.24)$$

in which all components grow at the same relative rate. The results (3.16) to (3.24) can be expressed concisely by plotting the asymptotic form for the total kinetic energy, $\overline{u_1^2} + \overline{u_2^2} + \overline{u_3^2}$, expressed as a power of c, the expansion ratio, against the geometrical parameter α. In figure 1 we plot γ against α, where γ is given by the relation:

$$\text{Total kinetic energy} \propto c^{\gamma}.$$

The point $\alpha = 0$ is special in that

$$\text{Total kinetic energy} \propto c(\ln c)^{-1},$$

but as has been explained above this is only an apparent singularity in behaviour.

The results can be further simplified in the case of initially isotropic turbulence, for the tensor spectrum functions can then be expressed in terms of an energy spectrum function $E(k)$, which is such that

$$\Omega_{ij}^{(0)}(\boldsymbol{\chi}) = \frac{E(\chi)}{4\pi\chi^2}(\chi^2\delta_{ij} - \chi_i\chi_j). \tag{3.25}$$

For $0 < \alpha < \frac{1}{2}$

$$\overline{u_2^2}, \overline{u_3^2} \to \frac{\alpha}{4\nu T} c^{1-2\alpha} \int_0^\infty \frac{E(l)}{l^2} e^{-\nu T l^2} dl. \tag{3.26}$$

For $\alpha = \frac{1}{2}$

$$\overline{u_2^2}, \overline{u_3^2} \to \frac{1}{8\nu T} \int_0^\infty \frac{E(l)}{l^2} e^{-\nu T l^2} dl. \tag{3.27}$$

For $\alpha = 0$

$$\overline{u_2^2}, \overline{u_3^2} \to \frac{1}{8\nu T} \frac{c}{\ln c} \int_0^\infty \frac{E(l)}{l^2} e^{-\nu T l^2} dl. \tag{3.28}$$

For $-1 < \alpha < 0$

$$\begin{aligned} \overline{u_2^2}, \overline{u_3^2} &\to \frac{c}{2} \iint_0^\infty \frac{l_2 E[(l_1^2 + l_2^2)^{\frac{1}{2}}]}{l_1^2 + l_2^2} \exp\left\{ -\nu T \left(l_1^2 + \frac{l_2^2}{\alpha} \right) \right\} dl_1\, dl_2 \\ &= \frac{c}{2} \sqrt{\left\{ \frac{-\alpha}{\nu T(1+\alpha)} \right\}} \int_0^\infty \frac{E(l)}{l} e^{-\nu T l^2/\alpha} \int_0^{\sqrt{\left\{\frac{\nu T(1+\alpha)}{-\alpha}\right\}}\, l} e^{y^2}\, dy\, dl. \end{aligned} \tag{3.29}$$

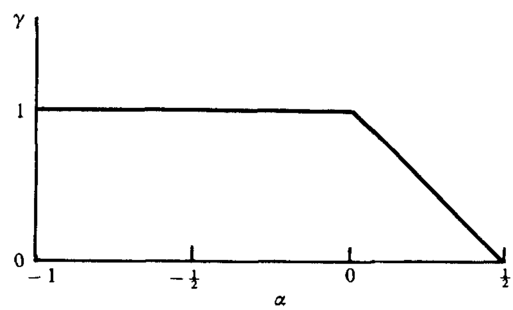

FIGURE 1. Growth of kinetic energy expressed as a power, γ, of the expansion ratio *vs* α.

For $\alpha = -1$

$$\overline{u_1^2}, \overline{u_2^2}, \tfrac{1}{2}\overline{u_3^2} \to \frac{\pi}{8} c \int_0^\infty E(l)\, e^{-\nu T l^2} dl. \tag{3.30}$$

Additional information concerning the orientation that is induced in the turbulence as $c \to \infty$ can be derived by considering the asymptotic form of the vorticity spectrum (3.12). This is such that the dominant term $\Omega_{11}(\mathbf{k}, c)$ is appreciable only when

$$k_1 \to 0 \quad \text{for} \quad \alpha > 0,$$
$$k_1, k_2 \to 0 \quad \text{for} \quad \alpha \leqslant 0.$$

By carrying out the Fourier transformations required to interpret this result in co-ordinate space we find that covariances become, in the limit, independent of separation in the r_1 direction for $\alpha > 0$, and independent of separation in both the r_1 and r_2 directions for $\alpha \leqslant 0$. Thus for $\alpha > 0$ we may say that the turbulence is truly two-dimensional, since the $\overline{u_1^2}$ component of intensity tends to zero (see (3.16)); for $-\frac{1}{2} < \alpha \leqslant 0$, the turbulence is not only two-dimensional, but covariances are functions of one space variable only, $\overline{u_1^2}$ still tending to zero (see (3.22)); for $-1 < \alpha \leqslant -\frac{1}{2}$ only the ratio $\overline{u_1^2}/\overline{u_2^2}$ tends to zero, and the result is a little weaker; for $\alpha = -1$ the turbulent intensity ceases to be oriented in a plane, though covariances remain functions of one space variable only.

The limiting structure of the velocity spectrum as $c \to \infty$ can also be stated. For the case $\alpha > 0$, we find that the dominant contribution tends to the form

$$\Phi_{ij}(\mathbf{k}, c) \to c^{2-2\alpha} \epsilon_{1im} \epsilon_{1jn} \frac{k_m k_n k_2^2}{(k_2^2 + k_3^2)^2} \frac{\partial^2 \Omega_{11}^{(0)}}{\partial k_2^2} (ck_1, 0, 0) \exp\left\{-\nu T\left(c^2 k_1^2 + \frac{k_2^2}{\alpha} + \frac{k_3^2}{1-\alpha}\right)\right\}.$$

Without attempting a rigorous definition, we may say that the spectrum function tends to a universal form where the integral parameter

$$\int_{-\infty}^{\infty} \frac{\partial^2 \Omega_{11}^{(0)}}{\partial k_2^2} (k_1, 0, 0)\, e^{-\nu T k_1^2}\, dk_1$$

is all that remains of the initial turbulent distribution.

For the case $\alpha < 0$, the asymptotic structure cannot be so neatly expressed. It is

$$\Phi_{ij}(\mathbf{k}, c) \to c^2 \epsilon_{1im} \epsilon_{1jn} \frac{k_m k_n + \delta_{mn} k_1^2}{k^4} \Omega_{11}^{(0)}(ck_1, c^{-\alpha} k_2, 0) \times \exp\left\{-\nu T\left(c^2 k_1^2 + \frac{c^{-2\alpha} k_2^2}{-\alpha} + \frac{k_3^2}{1-\alpha}\right)\right\}.$$

Since this takes its dominant values for k_1, k_2 and k_3 all tending to zero, a highly singular structure is approached. It is universal only in so far as energy is concentrated at zero wave-number.

4. Axisymmetric extension of an initially isotropic turbulent field in its final period of decay

In the previous section, attention has been concentrated on the asymptotic limit that applies as $c \to \infty$. We shall now consider the time-dependent solution for a particular case as an example of the way in which the asymptotic limit is approached. For simplicity we take the initial turbulence to be isotropic, and because our conditions refer specifically to weak turbulent fields we take the turbulence to be in its final period of decay. This may be described by the energy spectrum*

$$E(k) = k^4 e^{-k^2/k_0^2}, \tag{4.1}$$

where
$$\Phi_{ij}(\mathbf{k}, 0) = \frac{E(k)}{4\pi k^4}(k^2 \delta_{ij} - k_i k_j), \tag{4.2}$$

and k_0 is a characteristic wave-number.

We choose the case of axisymmetric distortion, for which $\alpha = \frac{1}{2}$, as our straining field. This will lead to a finite asymptotic solution for the total kinetic energy and

* See Batchelor (1953) and the subsequent paper Batchelor & Proudman (1956).

can be compared with the sudden strain solution given by B & P. We shall calculate, as suitable defining characteristics of the turbulent field, the ratios

$$\mu_1 = \frac{\overline{u_1^2(t)}}{\overline{u_1^2(0)}} = \frac{\int \Phi_{11}(\mathbf{k},t)\,d\mathbf{k}}{\int \Phi_{11}(\mathbf{k},0)\,d\mathbf{k}} \tag{4.3}$$

and

$$\mu_2 = \frac{\overline{u_2^2(t)+u_3^2(t)}}{\overline{u_2^2(0)+u_3^2(0)}} = \frac{\int \Phi_{22}(\mathbf{k},t)+\Phi_{33}(\mathbf{k},t)\,d\mathbf{k}}{\int \Phi_{22}(\mathbf{k},0)+\Phi_{33}(\mathbf{k},0)\,d\mathbf{k}}. \tag{4.4}$$

From (3.7), (3.12) and (3.13) we have that

$$\Phi_{11}(\mathbf{k},t) = \frac{(\chi_2^2+\chi_3^2)\,E(\chi)}{4\pi k^4}\exp\left[-\nu T\{(\chi_1^2-k_1^2)-2(\chi_2^2-k_2^2)-2(\chi_3^2-k_3^2)\}\right],$$

$$\Phi_{22}(\mathbf{k},t)+\Phi_{33}(\mathbf{k},t) = \left(\frac{c}{\chi^2}+\frac{\chi_1^2}{k^4c^3}\right)\frac{E(\chi)}{4\pi} \tag{4.5}$$

$$\times\exp\left[-\nu T\{(\chi_1^2-k_1^2)-2(\chi_2^2-k_2^2)-2(\chi_3^2-k_3^2)\}\right], \tag{4.6}$$

where

$$\chi^2 = \chi_1^2+\chi_2^2+\chi_3^2 \tag{4.7}$$

and

$$(\chi_1,\chi_2,\chi_3) = (ck_1, c^{-\frac{1}{2}}k_2, c^{-\frac{1}{2}}k_3). \tag{4.8}$$

On substituting (4.5) and (4.6) into (4.3) and (4.4) and after slight simplification we find that μ_1 and μ_2 may be written

$$\mu_1 = \frac{3}{4c^2}\int_{-1}^{1}\frac{(1-x^2)\,dx}{\{1-(1-c^{-3})\,x^2\}^2\{[1+2(c-1)\,\tau]+\tau(3-2c-c^{-2})\,x^2\}^{\frac{5}{2}}}, \tag{4.9}$$

$$\mu_2 = \tfrac{3}{8}c\int_{-1}^{1}\left[1+\frac{x^2}{c^6\{1-(1-c^{-3})\,x^2\}^2}\right]\frac{dx}{\{[1+2(c-1)\,\tau]+\tau(3-2c-c^{-2})\,x^2\}^{\frac{5}{2}}}, \tag{4.10}$$

where

$$\tau = \nu T k_0^2. \tag{4.11}$$

τ may be regarded as an inverse Reynolds number where the length scale, k_0^{-1}, is derived from the characteristic length scale of the initial turbulent spectrum, and the time scale, T, is derived from the mean rate of strain. It will be observed that the integrals (4.9) and (4.10) for μ_1 and μ_2 are functions of c and τ only. These integrals have been evaluated exactly and are displayed in graphical form in figures 2–5 for $\tau = 1, 0{\cdot}25, 0{\cdot}1, 0{\cdot}05$, as continuous functions of c. (Using the relation (3.2), these can readily be converted into relations between μ_1 (or μ_2) and time, t.) Also included are the relevant curves for $\mu_1(t)$, $\mu_2(t)$ that would apply in the absence of distortion; interpreted for convenience as $\mu_1'(c)$, $\mu_2'(c)$, these provide a measure of the relative action of viscosity in the absence of mean strain and are calculated from the relation

$$\mu_1' = \mu_2' = \frac{1}{(1+2\nu k_0^2 t)^{\frac{5}{2}}} = \frac{1}{(1+2\tau\ln c)^{\frac{5}{2}}}, \tag{4.12}$$

which is readily derived. The sudden strain approximation of B & P is of course the limiting case that applies when $\tau \to 0$.

We may also derive from (4.9) and (4.10) the asymptotic limits, as $c \to \infty$, ($\tau \neq 0$),

$$\mu_{1\infty} \to \frac{3}{2(1+\tau)^{\frac{3}{2}}} \frac{\ln c}{c^2}, \tag{4.13}$$

$$\mu_{2\infty} \to \frac{1}{8\tau(1+\tau)^{\frac{3}{2}}}. \tag{4.14}$$

These correspond to the results (3.16′), (3.17′–3.18′) obtained earlier, and of course (4.14) may be obtained directly from the latter result. In general terms we see that $\overline{u_1^2}$ decays more rapidly than it would in the absence of mean strain, while $\overline{u_2^2}+\overline{u_3^2}$ approaches a steady state in which the rate of growth due to straining is exactly balanced by the viscous decay.

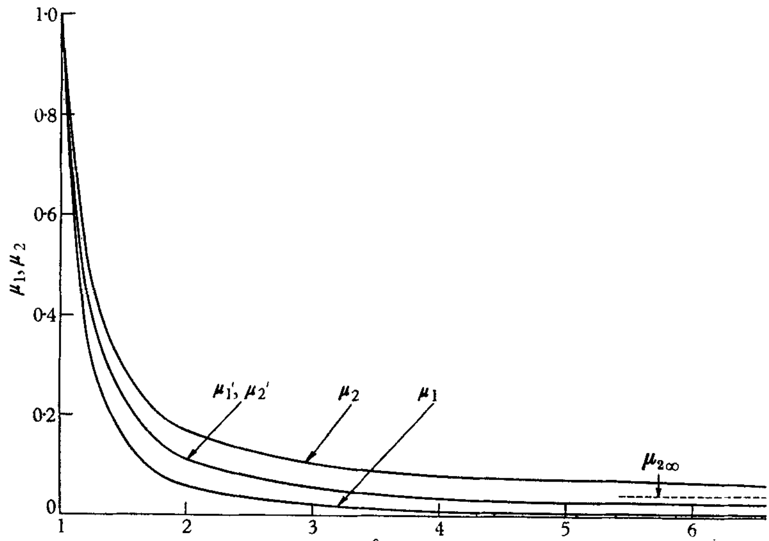

FIGURE 2. μ_1, μ_2 as functions of c; $\tau = 1$.

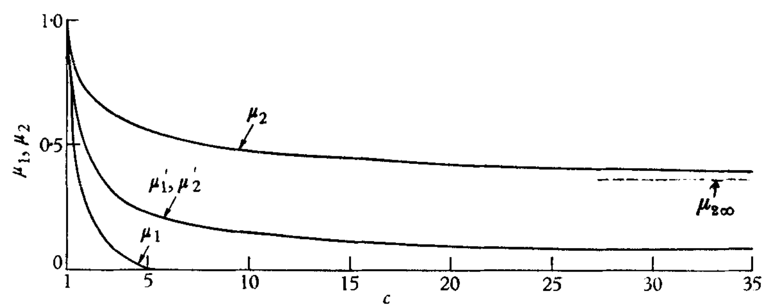

FIGURE 3. μ_1, μ_2 as functions of c; $\tau = 0{\cdot}25$.

5. Discussion

From the linearized treatment of strained turbulence given above and in particular from a consideration of the asymptotic behaviour as $c \to \infty$ (see § 3) it seems clear that in a general irrotational strain a balance cannot be achieved between interaction and viscous terms alone. The total energy associated with the turbulent components grows indefinitely and, as has been pointed out earlier, must ultimately reach the state at which non-linear turbulent effects are appreciable. This aspect of the problem is treated in some detail by Townsend (1956), Chap. 4, who considers the equilibrium structure attained by the turbulence when it is subjected to just the type of irrotational distortion represented by (3.1). [From an experimental point of view, irrotational distortions are the only

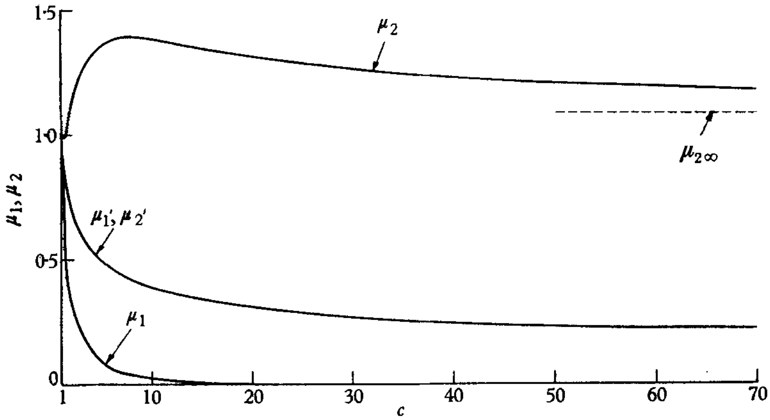

FIGURE 4. μ_1, μ_2 as functions of c; $\tau = 0{\cdot}1$.

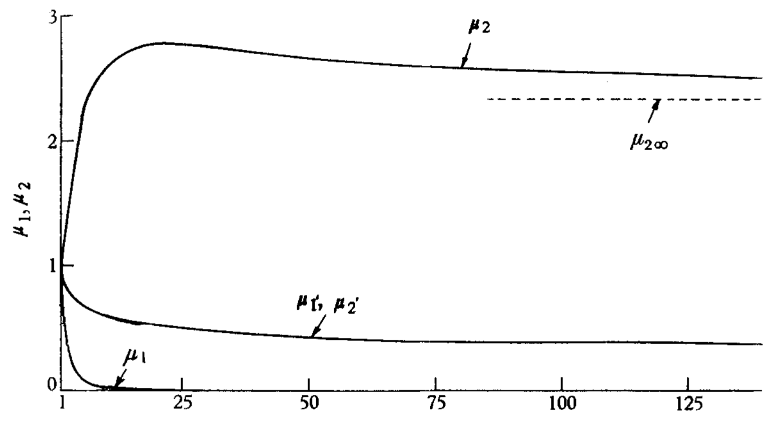

FIGURE 5. μ_1, μ_2 as functions of c; $\tau = 0{\cdot}05$.

type that can be set up instantaneously and so the restriction on generality that has been accepted above for reasons of expediency finds some justification in physical circumstances.] It had been hoped, when this work was started, to include inertial effects in the dynamical treatment, in the manner of P & R, and to use the non-inertial solution obtained in § 3 as a first approximation (from the point of view of structure) to the problem of maintained shear flow turbulence. However, despite the simplifications achieved by considering the turbulence to be two-dimensional, as described at the end of § 3, a dynamical treatment including non-linear terms proved to be virtually intractable.

A more particular application of the linearized solution is suggested by Ribner & Tucker (1952) who summarize certain experimental evidence concerning wind-tunnel turbulence. This refers to the reduction in relative energy of the residual turbulence that persists after screening when the air stream is passed through a rapid contraction. Although the analysis has not been given here it is a relatively simple matter to extend the solution (3.12) to cover the case there C_{ij} is a function of time; it may be supposed that a small element of fluid is subjected as it passes through the contraction to a time varying, but instantaneously uniform, contraction corresponding to the local value for the contraction at its particular station within the wind-tunnel. In this way the behaviour of the low energy turbulent field entering the contraction may be followed through the contraction, account being taken of viscosity as well as of local strain.

However, the calculations involved in an exact analysis would be formidable and the only practicable method would be to employ an approximate solution based on the treatment given in §4 as a crude model. A consideration of the Reynolds number for the turbulence entering the wind-tunnel contraction—based on the integral length scale and the root mean square intensity—for the cases quoted by Ribner & Tucker [11 in the case of McPhail (1944) and 7 in the case of Hall (1938)] shows that the hypothesis of initially weak turbulence is not unreasonable; for the experiments of Batchelor & Townsend (1948) show that the final period of decay is reached at a Reynolds number—based on the dissipation length scale—of about 5. What the results of §3 show is that an axisymmetric contraction is probably the most effective type of contraction for reducing the relative turbulent intensity.

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REFERENCES

Batchelor, G. K. 1953 *The Theory of Homogeneous Turbulence*. Cambridge University Press.

Batchelor, G. K. & Proudman, I. 1954 *Quart. J. Mech. Appl. Math.* **7**, 83.

Batchelor, G. K. & Proudman, I. 1956 *Phil. Trans.* A, **248**, 369.

Batchelor, G. K. & Townsend, A. A. 1948 *Proc. Roy. Soc.* A, **194**, 527.

Hall, A. A. 1938 *A.R.C. Tech. Report* no. 1842.

McPhail, D. C. 1944 *R.A.E. Report* no. Aero 1928.

Proudman, I. & Reid, W. H. 1954 *Phil. Trans.* A, **247**, 163.

Ribner, H. S. & Tucker, M. 1952 *N.A.C.A. Tech. Note* 2606.

Townsend, A. A. 1956 *The Structure of Turbulent Shear Flow*. Cambridge University Press.